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SUBGRID MODELING OF FILTRATION IN POROUS SELF-SIMILAR MEDIA

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Subgrid modeling of a filtration flow of a fluid in an inhomogeneous porous medium is considered. An expression for the effective permeability coefficient for the large-scale component of the flow is derived using the scale-invariance hypothesis. The model obtained is verified by numerical simulation of the complete problem.

Introduction. The inhomogeneity of a porous medium has a significant effect on filtration processes. Natural porous media are essentially inhomogeneous, which can be taken into account as follows [1, 2]. Large-scale (coarse) elements of the medium structure, for instance, large zones into which the medium can be divided and easily distinguished layers or interlayers are directly described by the model. Small-scale details of permeability and porosity distributions are unknown. They should be taken into account within the statistical approach, introducing effective parameters, such as porosity, permeability, etc.

Theoretical determination of effective parameters requires solving of the corresponding problems in media with random fields. In the present work, the effective parameters are determined using an refined Kolmogorov's scale-invariance hypothesis for three-dimensional media.

The ideas of scale invariance were used by Kolmogorov [3] to describe the scheme of a random process of successive fragmentation of particles (gold particles in auriferous gravels, rock particles, etc.). To take into account the intermittency of an inhomogeneous medium, one can use the ideas proposed by Kolmogorov in his paper on the local structure of turbulence of a viscous incompressible fluid at very high Reynolds numbers [4]. In this approach, the basic properties of the media are self-similarity, hierarchical spatial structurization, and power dependences. For porous sedimentary rocks, power dependences were found, and correlation functions of density and other parameters of the medium were measured [2]. The use of scale-invariance hypotheses similar to those proposed by Kolmogorov [4] allows one to obtain dependences close to experimental ones.

Kolmogorov's infinite multiplicative cascades [5] yield rather inhomogeneous "holed" (Cantor, fractal) sets [6], which are widely used in geophysical problems. Application of methods of the fractal theory requires the use of a geometric language, which is not sufficiently well mastered by many specialists working with porous inhomogeneous media. Meanwhile, the main ideas used in the present work are close to those in [4]. Another advantage of the approach used is the fact that the main parameters and functions of the theory of inhomogeneous media are close to those obtained in experiments.

1. Scale Invariance of a Porous Medium. A porous medium is described by a set of fields (porosity, permeability, coefficients in Hooke's law, etc.), which may depend on spatial coordinates and time. When these fields are measured, they are inevitably smoothed in space and time. The dependence of parameters on time is usually rather weak; therefore, spatial smoothing is most important. Let us consider the permeability field of the medium, which determines the filtration velocity of the fluid in the presence of a pressure gradient.

Let an incompressible fluid flow through a porous medium with a permeability coefficient $\epsilon(\boldsymbol{x})$. At low Reynolds numbers, the filtration velocity $\boldsymbol{v}$ and pressure p are related by the Darcy law $\boldsymbol{v} = \epsilon(\boldsymbol{x})\nabla p$. The condition of incompressibility $\operatorname{div} \boldsymbol{v} = 0$ yields the equation

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$$\frac{\partial}{\partial x_j}\Big(\epsilon(\boldsymbol{x})\frac{\partial}{\partial x_j}\,p(\boldsymbol{x})\Big) = 0. \tag{1}$$

Let the field of permeability be known. This means that it is measured at each point $\boldsymbol{x}$ as the fluid is pumped through a specimen of small size l_0. A random function of spatial coordinates $\epsilon(\boldsymbol{x})$ is considered as the limit of permeability $\epsilon(\boldsymbol{x}, l_0)$. As $l_0 \to 0$, we have $\epsilon(\boldsymbol{x}, l_0) \to \epsilon(\boldsymbol{x})$. The dependence $\epsilon(\boldsymbol{x}, l_0)$ on the scale l_0 can be considered as a factor that allows development of new approaches to studying randomly inhomogeneous media rather than the error of the measurement processes. To pass to a coarser grid l_1, one can smooth the resultant field $\epsilon(\boldsymbol{x}, l_0)$ using the scale $l_1 > l_0$. It is not clear, however, whether the field obtained is the true permeability that describes filtration in the range of scales (l_1, L), where L is the greatest scale. Generally speaking, this is not the case. To find permeability on a coarser grid, one has to repeat the measurements, pumping the fluid through larger specimens of size l_1. This procedure is necessary, since the fluctuations of permeability within the scale interval (l_0, l_1) have correlations with pressure fluctuations induced by them. They are found using Eq. (1). The search for the law of transformation of effective permeability with changing grid size is facilitated in media possessing scale invariance.

Similar to [4, 5], we consider a dimensionless field ψ equal to the ratio of permeability smoothed using two different scales l and l':

$$\psi(\boldsymbol{x}, l, l') = \varepsilon(\boldsymbol{x}, l')/\varepsilon(\boldsymbol{x}, l), \qquad l' < l$$

[$\varepsilon(\boldsymbol{x}, l)$ is the permeability $\epsilon(\boldsymbol{x}, l_0)$ smoothed over the scale l]. In what follows, we understand smoothing over the scale l as, for example, rejection of Fourier harmonics with wavenumbers $k \geqslant l$ in the expansion of the function examined. The hypothesis of statistical scale invariance is assumed to be valid for the relative fields $\psi(\boldsymbol{x}, l, l')$. According to this hypothesis, there is an interval of scales $l_0 < l < L$, where the correlation functions of the fields $\psi(\boldsymbol{x}, l, l')$ of all orders are invariant to the scale transform

$$\boldsymbol{x} \to K\boldsymbol{x}, \qquad l \to Kl, \qquad l' \to Kl'.$$

The corollaries of the scale-invariance hypothesis for the field $\psi(\boldsymbol{x}, l, l')$ are complicated by a large number of its arguments, but it is possible to find a simpler field that is uniquely related to $\psi(\boldsymbol{x}, l, l')$ and possesses all its properties.

By construction, the function ψ has the following property:

$$\psi(\boldsymbol{x}, l, l'') = \psi(\boldsymbol{x}, l, l')\psi(\boldsymbol{x}, l', l''). \tag{2}$$

For a scale l'' close to l', we expand the function $\psi(\boldsymbol{x}, l, l'')$ into a series in l'' at the point l':

$$\psi(\boldsymbol{x}, l, l'') = \psi(\boldsymbol{x}, l, l') + \frac{\partial\psi(\boldsymbol{x}, l, l')}{\partial l'}\,(l'' - l') + \ldots\ . \tag{3}$$

Similarly, we expand the function $\psi(\boldsymbol{x}, l', l'')$ into a series at the point l':

$$\psi(\boldsymbol{x}, l', l'') = \psi(\boldsymbol{x}, l', l') + \frac{\partial\psi(\boldsymbol{x}, l', l'')}{\partial l''}\Big|_{l''=l'}(l'' - l') + \ldots = 1 + \frac{\partial\psi(\boldsymbol{x}, l', l'y)}{l'\partial y}\Big|_{y=1}(l'' - l') + \ldots\ . \tag{4}$$

Here $y = l''/l'$. Substituting expansions (3) and (4) into Eq. (2), we obtain the following equation for $\psi(\boldsymbol{x}, l, l')$:

$$\frac{\partial\psi(\boldsymbol{x}, l, l')}{\partial l'} = \frac{1}{l'}\,\psi(\boldsymbol{x}, l, l')\varphi(\boldsymbol{x}, l'). \tag{5}$$

Here $\varphi(\boldsymbol{x}, l') = \partial\psi(\boldsymbol{x}, l', l'y)/l'\partial y|_{y=1}$. Equation (5) yields the relation

$$\varphi(\boldsymbol{x}, l) = \frac{\partial \ln\varepsilon(\boldsymbol{x}, l)}{\partial \ln l}. \tag{6}$$

We obtained the field $\varphi(\boldsymbol{x}, l)$, which determines the function $\epsilon(\boldsymbol{x}, l_0)$ but has the properties of simple scale invariance. All essential information about the porous medium is contained in statistical properties of the field $\varphi(\boldsymbol{x}, l)$. The medium is known if the statistical properties of the function $\varphi(\boldsymbol{x}, l)$ are defined. Scale-invariant fluctuations of the field φ can be observed only in a certain finite region of scales $l_0 < l < L$. The solution of Eq. (6) has the form

$$\epsilon(\boldsymbol{x}, l_0) = \varepsilon_0 \exp\Big(-\int_{l_0}^{L}\varphi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\Big). \tag{7}$$

For certainty, we assume that there are no inhomogeneities whose scale is larger than L. The coefficient ε_0 in Eq. (7) is assumed to be constant. We consider the correlation function (central moment) $\varphi(\boldsymbol{x}, l)$:

$$\Phi(\boldsymbol{x}, \boldsymbol{y}, l, l') = \langle \varphi(\boldsymbol{x}, l)\, \varphi(\boldsymbol{y}, l') \rangle_c = \langle \varphi(\boldsymbol{x}, l)\, \varphi(\boldsymbol{y}, l') \rangle - \langle \varphi(\boldsymbol{x}, l) \rangle \langle \varphi(\boldsymbol{y}, l') \rangle.$$

Hereinafter, the central moments are marked by the subscript c after the closing broken bracket. We assume that the medium is isotropic and statistically independent of the position of the origin of the coordinate system. Then, the correlation function considered depend only on the following arguments:

$$\Phi(\boldsymbol{x}, \boldsymbol{y}, l, l') = \Phi((\boldsymbol{x} - \boldsymbol{y})^2, l, l').$$

For simplicity, we use the same notation Φ in the right side. The statistical scale invariance of the field φ means that the following equality is valid for all positive K:

$$\Phi((\boldsymbol{x} - \boldsymbol{y})^2, l, l') = \Phi(K^2(\boldsymbol{x} - \boldsymbol{y})^2, Kl, Kl').$$

Choosing $K = 1/l$, we obtain

$$\Phi((\boldsymbol{x} - \boldsymbol{y})^2, l, l') = \Phi((\boldsymbol{x} - \boldsymbol{y})^2/l^2, l'/l). \tag{8}$$

It follows from this equation that the function Φ depends on two arguments only. If the field φ is not Gaussian, correlations of all orders are essential. Relations similar to (8) can readily be derived for them. These relations are not presented here, since the simplest scale-invariant model (logarithmically normal model) is used in numerical simulation. Note, natural extension of scale invariance (conformal symmetry) was considered in [7]. Conformal invariance allows one to limit the functional dependence of correlation functions of all orders more rigorously than the use of one scale invariance.

2. Logarithmically Normal Model for Permeability. In this work, we assume that the random field φ has the Gaussian distribution. In this case, the number of its independent correlation functions reduces to the first two functions. In the theory of probability, random Gaussian quantities are considered as the simplest objects. The permeability $\epsilon(\boldsymbol{x}, l_0)$ has a logarithmically normal distribution. Thus, the scale-invariant, logarithmically normal model is the simplest model in the class of scale-symmetric models.

Note, the following equality should be satisfied in this logarithmically normal model:

$$\langle \epsilon(\boldsymbol{x}, l_0) \rangle = \varepsilon_0. \tag{9}$$

For such fields as the porosity field, condition (9) follows from their physical essence. This condition is also valid for the field of permeability, since smoothing over large volumes is equivalent to statistical averaging in accordance with the ergodic hypothesis. Equality (9) shows that the constant ε_0 is equal to permeability smoothed over a large volume and differs from the effective permeability ε_{00} measured at the greatest scale, which is determined by pumping the fluid through a very large specimen of scale L.

Using Eqs. (7) and (9), we obtain

$$\Big\langle \exp\Big(- \int_{l_0}^{L} \varphi(\boldsymbol{x}, l_1) \frac{dl_1}{l_1} \Big) \Big\rangle = 1. \tag{10}$$

According to [5], the following formula is valid for an arbitrary nonrandom function $\theta(l)$ and Gaussian field $f(l)$:

$$\Big\langle \exp\Big(- i \int_{l}^{L} \theta(l_1) f(l_1)\, dl_1 \Big) \Big\rangle = \exp\Big(- i \int_{l}^{L} \theta(l_1) \langle f(l_1) \rangle\, dl_1 - \frac{1}{2} \int_{l}^{L} dl_1 \int_{l}^{L} dl_1\, \theta(l_1)\theta(l_2) \langle f(l_1) f(l_2) \rangle_c \Big). \tag{11}$$

Choosing $f(l) = \varphi(\boldsymbol{x}, l)$ and $\theta(l) = -i/l$, we obtain from Eqs. (10) and (11) the equation

$$\int_{l_0}^{L} \frac{dl_1}{l_1} \int_{l_0}^{L} \frac{dl_2}{l_2} \langle \varphi(\boldsymbol{x}, l_1) \varphi(\boldsymbol{x}, l_2) \rangle_c - 2 \int_{l_0}^{L} \langle \varphi(\boldsymbol{x}, l_1) \rangle \frac{dl_1}{l_1} = 0, \tag{12}$$

where $\langle \varphi \rangle$ is a constant, which follows from the homogeneity and scale invariance of $\varphi(\boldsymbol{x}, l)$. In a particular case of noncorrelated fluctuations of the field φ of different scales, the model is simplified. We choose the correlation function in the form

$$\Phi((\boldsymbol{x} - \boldsymbol{y})^2/l^2, l'/l) = \Phi_0 \exp(-(\boldsymbol{x} - \boldsymbol{y})^2/l^2) \delta(\ln l - \ln l'). \tag{13}$$

Then for $\boldsymbol{x} = \boldsymbol{y}$, Eq. (12) yields

$$\Phi_0 = 2\langle\varphi\rangle. \tag{14}$$

Formula (10) is valid for arbitrary limits of integration, since the exponent in (12) vanishes if Eq. (14) is satisfied.

We consider the correlation function ϵ at the points $\boldsymbol{x}$ and $\boldsymbol{x}+\boldsymbol{r}$. It follows from (7) that

$$\langle\epsilon(\boldsymbol{x}, l_0)\epsilon(\boldsymbol{x}+\boldsymbol{r}, l_0)\rangle = \Big\langle \varepsilon_0^2 \exp\Big[-\Big(\int\limits_{l_0}^{L} (\varphi(\boldsymbol{x}, l_1) + \varphi(\boldsymbol{x}+\boldsymbol{r}, l_1))\frac{dl_1}{l_1}\Big)\Big]\Big\rangle.$$

Choosing $f(l) = \varphi(\boldsymbol{x}, l) + \varphi(\boldsymbol{x}+\boldsymbol{r}, l)$ and $\theta(l) = -i/l$ in Eq. (11), we obtain

$$\langle\epsilon(\boldsymbol{x}, l_0)\epsilon(\boldsymbol{x}+\boldsymbol{r}, l_0)\rangle = \varepsilon_0^2 \exp\Big(-2\langle\varphi\rangle \ln\frac{L}{l_0} + \int\limits_{l_0}^{L}\frac{dl_1}{l_1}\int\limits_{l_0}^{L}\frac{dl_2}{l_2}\langle\varphi(\boldsymbol{x}, l_1)\varphi(\boldsymbol{x}+\boldsymbol{r}, l_2)\rangle_c\Big)$$

$$= \varepsilon_0^2 \exp\Big(-2\langle\varphi\rangle \ln\frac{L}{l_0} + \int\limits_{l_0}^{L}\frac{dl_1}{l_1}\int\limits_{l_0}^{L}\Phi\Big(\frac{r^2}{l_1^2}, \frac{l_1}{l_2}\Big)\frac{dl_2}{l_2}\Big). \tag{15}$$

Using (15), we estimate the correlation function ϵ from (13) for $r < L$, which is analogous to the formula for the correlation function of the energy-dissipation rate [5]

$$\langle\epsilon(\boldsymbol{x}, l_0)\epsilon(\boldsymbol{x}+\boldsymbol{r}, l_0)\rangle \approx C(r/L)^{-\Phi_0}. \tag{16}$$

The constant in (16) is defined by the expression $C = \varepsilon_0^2(L/l_0)^{-2\langle\varphi\rangle}\,\mathrm{e}^{-\Phi_0\gamma}$, where $\gamma = 0.57722$ is the Euler constant and l_0 is the minimum scale. For $r \gg L$, we have $\langle\epsilon(\boldsymbol{x}, l_0)\epsilon(\boldsymbol{x}+\boldsymbol{r}, l_0)\rangle \to \varepsilon_0^2$. This case is not interesting and is not further considered. In the case of conformal symmetry, similar estimates were obtained in [7] without using the assumption of the absence of correlation in terms of $\ln l$.

The constant C in Eq. (16) is not universal, since it is determined by an integral that extends outside the limits of the interval of scale invariance, whereas the exponent $\Phi_0 = 2\langle\varphi\rangle$ is universal, since it is proportional to the mean of the universal field φ. For sedimentary rocks, the value of the correlation function of density of sandstone is $2\langle\varphi\rangle \approx 0.3$ [2]. In theoretical studies, one may avoid using correlation functions with a delta-shaped dependence in terms of scale. This is not important, but the use of correlation functions significantly simplifies numerical simulation of such a field. Results of numerical experiments may be useful in solving problems for a wider range of real media.

3. Subgrid Model. We assume that the permeability ϵ varies within a wide range of scales; therefore, direct calculation of the pressure field from the equation is impossible or requires large computational resources. In the present paper, we show that Eq. (1) can be simplified to describe correctly the large-scale component of pressure only and contain no information about small-scale fluctuations by using a universal formula, if the small-scale component contains only the scale-invariant part of permeability fluctuations.

We represent the function of permeability $\epsilon(\boldsymbol{x}, l_0)$ as a sum of two components. The large-scale component is obtained by smoothing $\epsilon(\boldsymbol{x}, l_0)$, and the small-scale (subgrid) component is the difference $\varepsilon' = \epsilon(\boldsymbol{x}, l_0) - \varepsilon_l$. By virtue of Eq. (10), which is valid for arbitrary limits of integration in the model considered, the large-scale permeability ε_l is also obtained by statistical averaging of ϵ over small-scale fluctuations. The statistical distribution for p is determined by Eq. (1) if the distribution for ϵ is known. We determine the large-scale (ongrid) component of pressure as a conventional mean $p_l(\boldsymbol{x}) = \langle p(\boldsymbol{x})\rangle_l$. We consider the statistical mean of $p(\boldsymbol{x})$ — solutions of Eq. (1) in which the permeability coefficient has a fixed large-scale part ε_l but an arbitrary small-scale component ε'. The subgrid component of pressure $p' = p - p_l$ is of no interest, but it cannot be neglected in the filtration equation

$$\frac{\partial}{\partial x_j}\Big(\varepsilon_l(\boldsymbol{x})\,\frac{\partial}{\partial x_j}\,p_l(\boldsymbol{x}) + \Big\langle\varepsilon'(\boldsymbol{x})\,\frac{\partial}{\partial x_j}\,p'(\boldsymbol{x})\Big\rangle_l\Big) = 0, \tag{17}$$

since the second expression in this equation may be significant. The form of this term is further determined from the subgrid model.

Using the hypothesis of scale invariance for the medium, we obtain an approximate gradient model for the subgrid term. We use the Landau–Lifshits approach [8] where the effective dielectric permeability of the mixture was calculated under simplifying assumptions on the smallness of fluctuations and their spatial scale [8, Sec. 9].

The equations, simplifying assumptions, and model obtained in the present work are similar to those in [8]. In deriving the subgrid formula, it is assumed that subgrid fluctuations are small in amplitude and have a small spatial scale. Numerical verification revealed good agreement with model calculations, and we give some assumptions on the reasons for this correspondence and the form of the effective parameter of expansion in a more complete theory at the end of this Section.

We estimate the second term in (17). Subtracting Eq. (17) from Eq. (1), we obtain the equation for the subgrid component of pressure:

$$\frac{\partial}{\partial x_j}\Big(\varepsilon'(\boldsymbol{x})\,\frac{\partial}{\partial x_j}\,p_l(\boldsymbol{x}) + \varepsilon(\boldsymbol{x},l)\,\frac{\partial}{\partial x_j}\,p'(\boldsymbol{x}) + \varepsilon'(\boldsymbol{x})\,\frac{\partial}{\partial x_j}\,p'(\boldsymbol{x}) - \Big\langle \varepsilon'(\boldsymbol{x})\,\frac{\partial}{\partial x_j}\,p'(\boldsymbol{x})\Big\rangle_l\Big) = 0. \tag{18}$$

The method for obtaining the subgrid model is similar to that used in the method of the renormalization group [9]. The subgrid equation is solved using the theory of perturbations, and the resultant solution is substituted into the ongrid equation. This procedure is repeated many times for larger separating scales. As a result, equations for transformation of parameters of the subgrid model with variation of the separating scale are obtained. The model similar to that obtained in [8] yields the law of transformation of parameters with a small variation of the separating scale.

Retaining terms of the first order with respect to ε' and p' in Eq. (18), we obtain

$$\frac{\partial}{\partial x_j}\Big(\varepsilon'(\boldsymbol{x})\,\frac{\partial}{\partial x_j}\,p_l(\boldsymbol{x}) + \varepsilon(\boldsymbol{x},l)\,\frac{\partial}{\partial x_j}\,p'(\boldsymbol{x})\Big) = 0.$$

Assuming that $\varepsilon(\boldsymbol{x},l)$ and $\partial p_l(\boldsymbol{x})/\partial x_j$ change slowly as compared to ε' and p', we can write

$$\Delta p'(\boldsymbol{x}) = -\varepsilon(\boldsymbol{x},l)^{-1}\nabla\varepsilon'(\boldsymbol{x})\nabla p_l(\boldsymbol{x}).$$

The solution of this equation can be represented as

$$p'(\boldsymbol{x}) = -\int G(\boldsymbol{x}-\boldsymbol{x}')\,\frac{\nabla\varepsilon'(\boldsymbol{x}')\nabla p_l(\boldsymbol{x}')}{\varepsilon(\boldsymbol{x}',l)}\,d\boldsymbol{x}', \tag{19}$$

where $G(\boldsymbol{x}-\boldsymbol{x}')$ is the Green function. Using the expression obtained for the component of subgrid pressure (19), we estimate the second term in Eq. (17):

$$-\Big\langle \varepsilon'(\boldsymbol{x})\frac{\partial}{\partial x_j}\int G(\boldsymbol{x}-\boldsymbol{x}')\,\frac{1}{\varepsilon(\boldsymbol{x}',l)}\,\frac{\partial}{\partial x_i'}\,\varepsilon'(\boldsymbol{x}')\,\frac{\partial}{\partial x_i'}\,p_l(\boldsymbol{x}')\,d\boldsymbol{x}'\Big\rangle_l$$

$$= \Big\langle \varepsilon'(\boldsymbol{x})\int \frac{\partial}{\partial x_j'}\,G(\boldsymbol{x}-\boldsymbol{x}')\,\frac{1}{\varepsilon(\boldsymbol{x}',l)}\,\frac{\partial}{\partial x_i'}\,\varepsilon'(\boldsymbol{x}')\,\frac{\partial}{\partial x_i'}\,p_l(\boldsymbol{x}')\,d\boldsymbol{x}'\Big\rangle_l$$

$$= \int \Big\langle \varepsilon'(\boldsymbol{x})\,\frac{\partial}{\partial x_j'}\,G(\boldsymbol{x}-\boldsymbol{x}')\,\frac{1}{\varepsilon(\boldsymbol{x}',l)}\,\frac{\partial}{\partial x_i'}\,\varepsilon'(\boldsymbol{x}')\,\frac{\partial}{\partial x_i'}\,p_l(\boldsymbol{x}')\Big\rangle_l\,d\boldsymbol{x}'.$$

Since $\varepsilon(\boldsymbol{x},l)$ and $\partial p_l(\boldsymbol{x})/\partial x_j$ change slowly as compared to ε' and p', we can use the theorem of the mean to obtain

$$\int \Big\langle \varepsilon'(\boldsymbol{x})\,\frac{\partial}{\partial x_j'}\,G(\boldsymbol{x}-\boldsymbol{x}')\,\frac{\partial}{\partial x_i'}\,\varepsilon'(\boldsymbol{x}')\Big\rangle_l\,d\boldsymbol{x}'\,\frac{1}{\varepsilon(\boldsymbol{x},l)}\,\frac{\partial}{\partial x_i}\,p_l(\boldsymbol{x})$$

$$\approx -\int G(\boldsymbol{x}-\boldsymbol{x}')\,\frac{\partial}{\partial x_j'}\,\frac{\partial}{\partial x_i'}\langle\varepsilon'(\boldsymbol{x})\varepsilon'(\boldsymbol{x}')\rangle_l\,d\boldsymbol{x}'\,\frac{1}{\varepsilon(\boldsymbol{x},l)}\,\frac{\partial}{\partial x_i}\,p_l(\boldsymbol{x}).$$

The correlation function $\langle\varepsilon'(\boldsymbol{x})\varepsilon'(\boldsymbol{x}')\rangle = f((\boldsymbol{x}-\boldsymbol{x}')^2)$ is invariant with respect to the origin and isotropic; therefore, we have $(\partial/\partial x_j')(\partial f/\partial x_i') = \delta_{ij}\,\Delta f/D$, where D is the dimensionality of the space

$$-\int \delta(\boldsymbol{x}-\boldsymbol{x}')\,\frac{\delta_{ij}}{D}\,\langle\varepsilon'(\boldsymbol{x})\varepsilon'(\boldsymbol{x}')\rangle_l\,d\boldsymbol{x}'\,\frac{1}{\varepsilon(\boldsymbol{x},l)}\,\frac{\partial}{\partial x_i}\,p_l(\boldsymbol{x}) = -\frac{1}{D}\,\langle\varepsilon'(\boldsymbol{x})\varepsilon'(\boldsymbol{x})\rangle_l\,\frac{1}{\varepsilon(\boldsymbol{x},l)}\,\frac{\partial}{\partial x_j}\,p_l(\boldsymbol{x}).$$

We assume that the dependence $\varepsilon(\boldsymbol{x})$ contains all scales in the interval $(l-\delta l, L)$ and is constructed in accordance with formula (7). In this case, $\varepsilon'(\boldsymbol{x})$ can be determined as follows:

$$\varepsilon'(\boldsymbol{x}) = \varepsilon(\boldsymbol{x},l-\delta l) - \varepsilon(\boldsymbol{x},l) = \varepsilon_0\Big[\exp\Big(-\int_{l-\delta l}^{L}\varphi(\boldsymbol{x},l_1)\,\frac{dl_1}{l_1}\Big) - \exp\Big(-\int_{l}^{L}\varphi(\boldsymbol{x},l_1)\,\frac{dl_1}{l_1}\Big)\Big].$$

Neglecting second-order terms, we obtain

$$\varepsilon'(\boldsymbol{x}) \simeq -\varepsilon(\boldsymbol{x}, l) \int\limits_{l-\delta l}^{l} \varphi(\boldsymbol{x}, l_1)\, \frac{dl_1}{l_1}.$$

Then, we have

$$\langle \varepsilon'(\boldsymbol{x})\varepsilon'(\boldsymbol{x})\rangle_{lc} = \varepsilon(\boldsymbol{x}, l)^2 \int\limits_{l-\delta l}^{l} \int\limits_{l-\delta l}^{l} \langle \varphi(\boldsymbol{x}, l_1)\varphi(\boldsymbol{x}, l_2)\rangle_{lc}\, \frac{dl_1}{l_1}\, \frac{dl_2}{l_2}.$$

The central and initial moments for $\varepsilon'(\boldsymbol{x})$ coincide, since $\langle \varepsilon'(\boldsymbol{x})\rangle = 0$. For the correlation function (13), the delta-correlation in terms of the scale logarithm yield the equality

$$\langle \varphi(\boldsymbol{x}, l_1)\varphi(\boldsymbol{x}, l_2)\rangle_{lc} = \langle \varphi(\boldsymbol{x}, l_1)\varphi(\boldsymbol{x}, l_2)\rangle_{c}.$$

Substituting this solution into the equation of filtration and using formula (13) for the Gaussian field φ, we obtain the following expression for the subgrid term in (17):

$$-(\Phi_0 \delta l/(lD))\nabla[\varepsilon(\boldsymbol{x}, l)\nabla p_l(\boldsymbol{x})]. \qquad (20)$$

This formula is similar to the expression for dielectric permeability of the mixture obtained in [8]. Substituting (20) into (17), we obtain the expression for effective permeability arising in the ongrid equation

$$\varepsilon(\boldsymbol{x}, l) = \varepsilon_{0l} \exp\Big(-\int\limits_{l}^{L} \varphi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\Big) = \frac{\varepsilon_{0l}}{\varepsilon_0}\, \varepsilon_l(\boldsymbol{x}, l), \qquad (21)$$

where ε_{0l} depends on the scale only and satisfies the differential equation

$$\frac{d \ln \varepsilon_{0l}}{d \ln l} = -\frac{\Phi_0}{D}.$$

The solution of this equation shows that the constant in ε_{0l} has a power dependence on the subgrid scale:

$$\varepsilon_{0l} = \varepsilon_{00}(l/L)^{-\Phi_0/D}. \qquad (22)$$

Thus, ongrid pressure is described by the equation

$$\frac{\varepsilon_{0l}}{\varepsilon_0}\, \frac{\partial}{\partial x_j}\Big(\epsilon(\boldsymbol{x}, l)\, \frac{\partial}{\partial x_j}\, p_l(\boldsymbol{x})\Big) = 0. \qquad (23)$$

In Eq. (23), the influence of the subgrid term is taken into account by a coefficient $\varepsilon_{0l}/\varepsilon_0$ constant in space. This influence is significant in the expression for filtration velocity and also if Eq. (1) contains a term with a derivative in time. If a coarse grid is used to simulate a flow through a fractal medium, effective permeability should be multiplied by a constant factor in accordance with Eq. (22).

Formula (22) contains two constants: ε_{00} and ε_{0l}. The mean flow velocity in a porous medium is $\langle \boldsymbol{v}\rangle = \varepsilon_{00}\nabla\langle p\rangle$. The constant ε_{0l} is used in the subgrid model. The constants ε_{00} and ε_{0l} are different; hence, subgrid fluctuations of pressure are significant. The difference between the constants ε_{00} and ε_0 should also be taken into account. The constant ε_0 is obtained by averaging Eq. (7) over fluctuations of all scales. The constant ε_{00} characterizes the relationship between the mean velocity and mean pressure gradient.

The power exponent in formula (22) is small for high values of D. This allows us to assume that the parameter Φ_0/D can be used as a small parameter in the theory of perturbations. Formula (20) yield a gradient model. The exact expression for subgrid terms should contain some nonlocal integral expressions. To refine the model, one has to take into account the next terms of expansion in Eq. (17). Formula (22) is verified by numerical simulation in Sec. 4.

4. Numerical Simulation. We solve the problem of filtration in an inhomogeneous porous medium in a cube with a rib L_0. A constant pressure is set at the faces $y = 0$ and $y = L_0$: $p(x, y, z)|_{y=0} = p_1$ and $p(x, y, z)|_{y=L_0} = p_2$ ($p_1 > p_2$). The pressure at the other faces of the cube is set by the linear dependence in terms of y: $p = p_1 + (p_2 - p_1)y/L_0$. The main filtration flow is directed along the y axis. Fluctuations of porosity induce fluctuations of the magnitude and direction of filtration velocity.

For numerical calculation of the flow, we pass to dimensionless variables in Eq. (1). All lengths are measured in units of L_0, and the unit of pressure difference is $p_1 - p_2$. Permeability is measured in units of ε_0. Thus, it is sufficient to solve the problem for $\varepsilon_0 = 1$ in a unit cube with a unit pressure difference.

First, we calculate the field of permeability. The integral in Eq. (7) is replaced by a finite-difference formula in which it is convenient to pass to logarithms with the base 2:

$$\varepsilon_l(\boldsymbol{x}) = \exp\Big(-\frac{1}{\log_2 e}\int\limits_{\log_2 l}^{\log_2 L_0} \varphi(\boldsymbol{x},\tau)\,d\tau\Big) = 2^{-\int_{\log_2 l}^{0} \varphi(\boldsymbol{x},\tau)\,d\tau} \approx 2^{-\sum_{i=-8}^{0} \varphi(\boldsymbol{x},\tau_i)\delta\tau}. \tag{24}$$

Here $l_i = 2^{\tau_i}$ and $\delta\tau$ is the step of discretization in terms of the scale logarithm. We use a $256 \times 256 \times 256$ grid with respect to spatial variables; $\delta\tau = 1$ and $\tau_i = 0, -2, -3, \ldots, \log_2(1/256) = -8$. To calculate φ, we use the correlation function (13) to obtain

$$\langle \varphi(\boldsymbol{x}, l_i)\, \varphi(\boldsymbol{y}, l_j)\rangle_c = (\Phi_0/\ln 2)\exp(-(\boldsymbol{x}-\boldsymbol{y})^2/2^{2\tau_i})\delta_{ij},$$

where the constant $\Phi_0 = 2\langle\varphi\rangle$ should be chosen from experimental data for natural porous media. According to [2], $\Phi_0 \approx 0.3$. The functions on the grid are matrices. The structure of the correlation matrix allows us to represent it in the form of a direct product of four matrices of lower dimensionality and apply the algorithm "along rows and columns" for numerical simulation [10]. The delta-correlation in the scale logarithm means that the field $\varphi(\boldsymbol{x}, l_i)$ is generated independently on each scale l_i. The total power exponent in (24) is summed over statistically independent layers.

In the present calculations, two upper and three lower layers are left empty, i.e., the field φ in these layers is equal to zero. Two empty upper layers indicate that the scale of the largest fluctuations of permeability is $L = 1/8$. This allows us to replace the approximate-probable mean quantities by space-averaged values. Three lower layers are also left empty, which is conditioned by the requirement that the difference problem considered should provide a good approximation of Eq. (1) at all scales.

In particular problems, the scales L and l_0 can take different values. They are not specified in the present work, since the objective is to find a universal subgrid model and its universal power exponents [such as Φ_0/D in Eq. (22)]. For an approximate calculation, a certain limited number of layers can be used. In our case, there are four layers. Figure 1 shows the change in permeability with increasing contribution of smaller and smaller fluctuations. The corresponding field of permeability for different scales in the mid-section $z = 1/2$ is given. In accordance with the procedure of deriving the subgrid formula, we have to solve numerically the complete problem and perform probability averaging over small-scale fluctuations to verify the formula. As a result, we obtain a subgrid term, which can be compared with the theoretical expression. Probability averaging requires multiple solution of the complete problem with a prescribed large-scale component of permeability but a random subgrid component with subsequent averaging with respect to the latter. In the present work, we performed a more economic variant of verification based on the power dependence of the total flow rate on the ratio of the maximum and minimum scales in the ongrid region in calculating permeability by Eq. (7) if the contribution of the subgrid region is ignored. To demonstrate this, we perform the following transformations of the formulas obtained. Effective permeability (21) should yield the true velocity in the region (L, l). In particular, the total flow rate of the fluid through the specimen should coincide with the true value regardless of the scale of truncation l.

For the mean flow rate of the fluid through a cross section, we have

$$-\Big\langle \varepsilon_{0l}\exp\Big(-\int\limits_{l}^{L}\varphi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\Big)\nabla p\Big\rangle = Q = \text{const}.$$

The right side of this equality is independent of l. Taking into account Eq. (22), we obtain

$$\varepsilon_{0l}\Big\langle \exp\Big(-\int\limits_{l}^{L}\varphi(\boldsymbol{x}, l_1)\,\frac{dl_1}{l_1}\Big)\nabla p\Big\rangle = \varepsilon_{00}\Big(\frac{l}{L}\Big)^{-\Phi_0/D}\Big\langle \exp\Big(-\int\limits_{l}^{L}\varphi(\boldsymbol{x}, l_1)\,\frac{dl_1}{l_1}\Big)\nabla p\Big\rangle = -Q.$$

Hence, we have

$$\Big\langle \exp\Big(-\int\limits_{l}^{L}\varphi(\boldsymbol{x}, l_1)\,\frac{dl_1}{l_1}\Big)\nabla p\Big\rangle = -\frac{Q}{\varepsilon_{00}}\Big(\frac{l}{L}\Big)^{\Phi_0/D}.$$

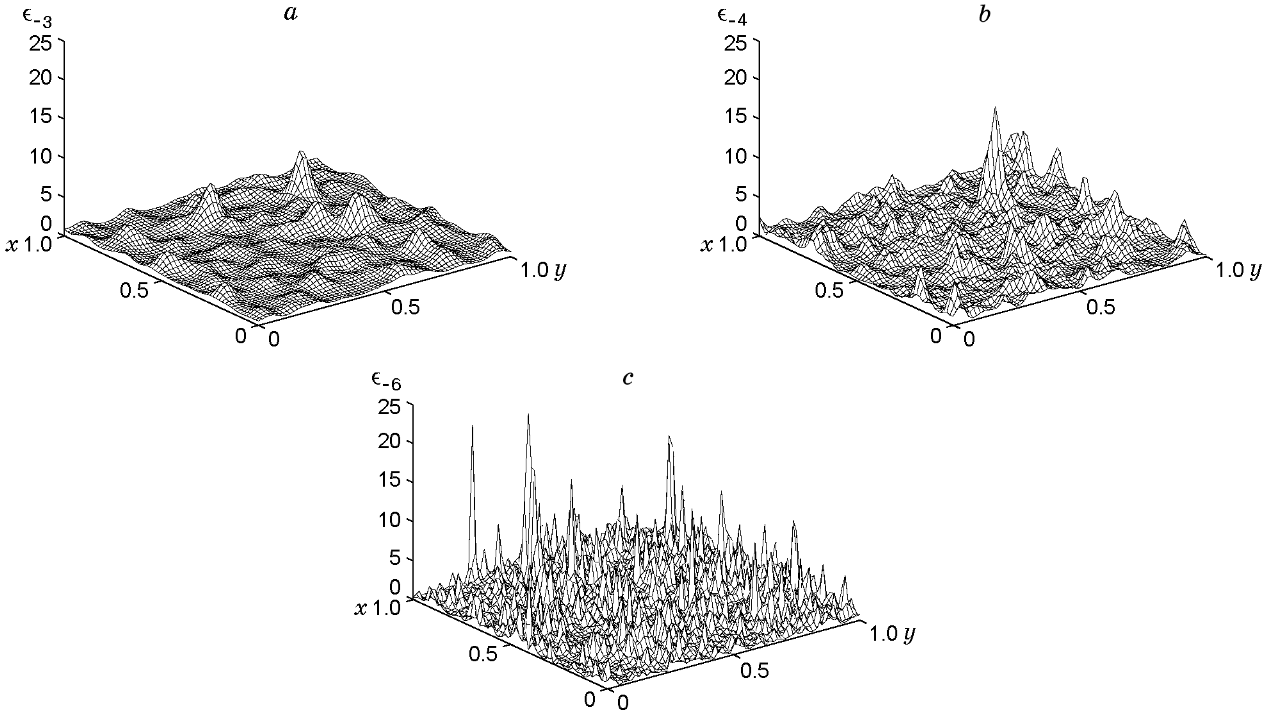

Fig. 1. Variation of permeability for $z = 1/2$ and $\epsilon_{-3}(\boldsymbol{x}) = 2^{-\sum\limits_{i=-3}^{-2} \varphi(\boldsymbol{x},\tau_i)}$ (a), $\epsilon_{-4}(\boldsymbol{x}) = 2^{-\sum\limits_{i=-4}^{-2} \varphi(\boldsymbol{x},\tau_i)}$ (b), and $\epsilon_{-6}(\boldsymbol{x}) = 2^{-\sum\limits_{i=-6}^{-2} \varphi(\boldsymbol{x},\tau_i)}$ (c).

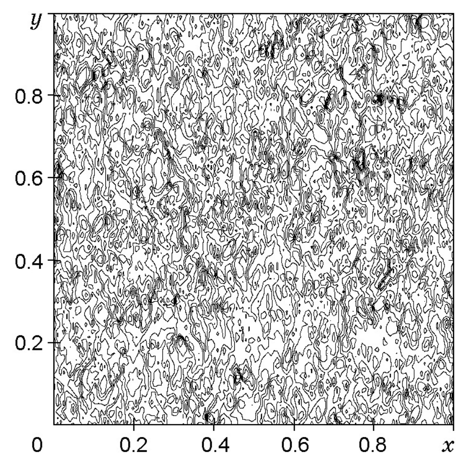

Fig. 2. Velocity isolines in the cross section $z = 1/2$.

In accordance with the Darcy law written at the scale $L_0 \gg L$, the total flow rate is

$$Q = \varepsilon_{00}(p_1 - p_2)/(y_2 - y_1).$$

In the present calculations, we assume that the scale L_0 is rather large as compared to L, and the probable mean can be replaced by space averaging (ergodic hypothesis). Thus, we have to verify numerically the formula

$$\Big\langle \exp\Big(- \int\limits_{l}^{L} \varphi(\boldsymbol{x}, l_1) \, \frac{dl_1}{l_1} \Big) \nabla p \Big\rangle = \frac{p_2 - p_1}{y_2 - y_1} \Big(\frac{l}{L} \Big)^{\Phi_0/D},$$

where the mean is understood as the space-averaged.

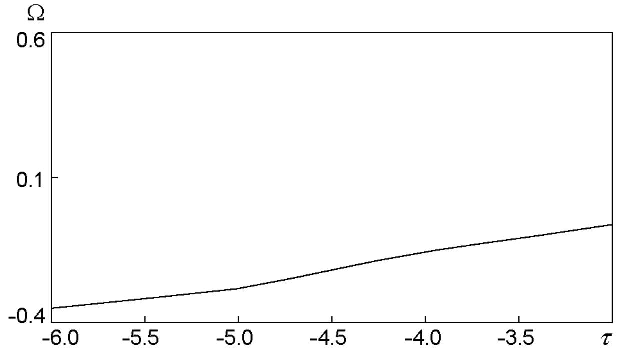

Fig. 3

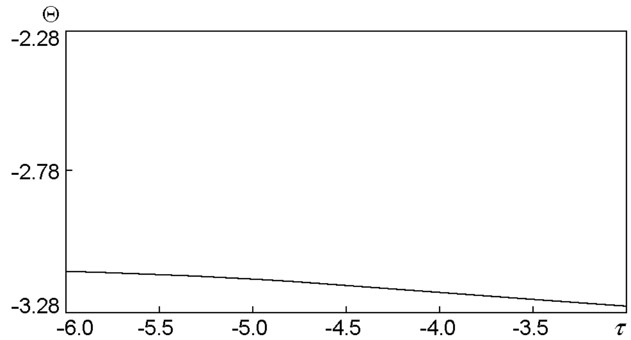

Fig. 4

Fig. 3. Dependence $\Omega(\tau)$ for $p_1 - p_2 = 1$ and $y_2 - y_1 = 1$.
Fig. 4. Dependence $\Theta(\tau)$.

Then, the grid analog of the dimensionless equation (1) is solved numerically. An iterative method combined with the fast Fourier transform and second-order sweep method is used [11]. Figure 2 shows the velocity isolines for the case where the layers numbered (-6)–(-3) over the scale are filled. Then we obtain the left side of Eq. (4) using the numerical solution of Eq. (1), where fluctuations of the smallest scale are $\varepsilon_{-3}, \ldots, \varepsilon_{-6}$. It was found theoretically that the plot should have the form of a straight line (since a power dependence was obtained) with a slope $\Phi_0/D = 0.1$ to the abscissa axis. The result of numerical verification is plotted in Fig. 3, where

$$\Omega = \log_2 \Big\langle \exp\Big(- \int_l^L \varphi(\boldsymbol{x}, l_1)\, \frac{dl_1}{l_1}\Big)\, \frac{\nabla p}{p_2 - p_1} \Big\rangle$$

is the flow rate of the fluid. The theoretical results are in rather good agreement with numerical data. Figure 4 shows the dependence of $\Theta = \log_2 [\langle(\nabla p)^2\rangle/(p_2 - p_1)^2]$ on τ. Obviously, this is also a power dependence, which testifies to scale invariance of the pressure-gradient field.

5. Discussion of Calculation Results. An analysis of the plots of permeability and filtration velocity shows that they are close to the results of numerical experiments with the use of the multifractal technique. Within the approach used in the present work, multifractals are obtained if the minimum scale l_0 tends to zero. In this case, rather irregular fields are obtained, which differ from zero on the Cantor-type set (see Fig. 1). One has to use the language of geometry to describe these fields. The numerical experiment on the Cantor sets is performed using the methodology of the percolation theory. A geometric set of the carpet type with bonds of different scales is usually constructed (for example, the Serpinskii carpet). These bonds can serve to transport a fluid, heat, an admixture, etc. In solving probability problems on fractals, singular distributions of probability are obtained, and then envelopes are constructed (see, e.g., [12]) for which differential equations are derived. In the present work, the minimum scale remains finite; therefore, there are no singularities. The Cantor sets do not emerge, and the entire analysis is within the limits of the apparatus of differential equations and the theory of random processes. The special feature of the problem is the use of the scale-invariance hypothesis. The main objects of the theory are fields whose properties can be measured directly (at least, in principle).

The simplest model considered within the framework of the present approach is the scale-invariant, logarithmically normal model of the carrier medium. The logarithmically normal model was criticized for several reasons. For example, the logarithmically normal distribution does not satisfy the conditions of the Carlemann theory and, hence, it is not determined by its moments. It is argued in some papers that the use of the logarithmically normal model does not involve these difficulties [13]. The method used in the present work is applicable for a wider class of models that the logarithmically normal model. It is assumed that the power dependences obtained are also typical of more generic cases.

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REFERENCES

1. M. I. Shvidler, *Statistical Hydrodynamics of Porous Media* [in Russian], Nedra, Moscow (1985).
2. M. Sahimi, "Flow phenomena in rocks: from continuum models, to fractals, percolation, cellular automata, and simulated annealing," *Rev. Mod. Phys.*, **65**, 1393–1534 (1993).
3. A. N. Kolmogorov, "Logarithmically normal law of particle-size distribution during fragmentation," *Dokl. Akad. Nauk SSSR*, **31**, 99–101 (1941).
4. A. N. Kolmogorov, "A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number," *J. Fluid Mech.*, **13**, 82–85 (1962).
5. A. S. Monin and A. M. Yaglom, *Statistische Hydromechanik.* 2, Verlag–Nauka, Moskau (1965).
6. B. B. Mandelbrot, *The Fractal Geometry of Nature*, Freeman, San Francisco (1983).
7. G. Kuz'min and O. Soboleva, "Conformal symmetric model of the porous media," *Appl. Math. Lett.*, **14**, 783–788 (2001).
8. L. D. Landau and E. M. Lifshits, *Course of Theoretical Physics*, Vol. 8: *Electrodynamics of Continuous Media*, Pergamon Press, Oxford-Elmsford, New York (1984).
9. D. Forster, D. R. Nelson, and M. J. Stephen, "Large-distance and long-time properties of a randomly stirred fluid," *Phys. Rev.*, **16**, 732–749 (1977).
10. V. A. Ogorodnikov and S. M. Prigarin, *Numerical Modeling of Random Processes and Fields: Algorithms and Applications*, Kluwer, Utrecht (1996).
11. G. I. Marchuk, *Methods of Computational Mathematics* [in Russian], Nauka, Moscow (1989).
12. Ben O'Shaughnessy and I. Procaccia, "Diffusion on fractals," *Phys. Rev.*, **13**, No. 5, 3073–3083 (1985).
13. G. M. Molchan, "Turbulent cascades: Multifractal characteristics," *Vychisl. Seismolog.*, **29**, 155–167 (1997).